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COMPARATIVE ANALYSIS OF VARIOUS METHODS OF ESTIMATING THE PARAMETERS IN THE PROBLEM OF V.G. KURT

L.S. Gurin, N.P. Ivanova, V.S. Mokrov, and K.A. Tsoy

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16. Abstract The article discusses various methods for estimating the parameters of particle fluxes representing a Poisson process. The particles arrive, through several filters, at a common counter which transmits the arrival time of each eighth particle. The mixing of the particles reduces the accuracy with which the parameters are determined. The efficiency of the estimation methods considered is compared, by means of a theoretical analysis and Monte Carlo simulation models using electronic digital computers. The results of the study are applied to the solution of concrete problems.

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COMPARATIVE ANALYSIS OF VARIOUS METHODS OF EVALUATING THE PARAMETERS IN THE PROBLEM OF V.G. KURT

L.S. Gurin, N.P. Ivanova, V.S. Mokrov and K.A. Tsoy

Section 1. Formulation of Problem /3*

Fluxes (of pulses, particles) with different intensities are observed. Each flux is isolated by an appropriate filter. Without loss of generality we will assume that the number of filters is four. It is assumed that the fluxes represent Poisson processes (in particular, the simplest Poisson processes) with parameters $\lambda_1,\ldots,\lambda_4$, which must be estimated. The filters are connected sequentially for equal time intervals to (t_o = 12 sec). The number of points (pulses, particles) is counted by one counter-adder, which accumulates the pulses from each connected filter and transmits the instants at which the number of points is a multiple of some number n (in practice, n = 8; below, we will consider only this case). The time interval $4t_o$ in which all filters were connected will be called a cycle. Thus, the result of the measurements represents a sequence of time instants t_j ($j = 1,\ldots,N$) in the interval T consisting of a specific number of cycles $T = 4mt$.

The following problems must be solved:

a. The unknown parameters λ_i ($i = 1,2,3,4$) must be estimated.

b. The parameters must be determined as a function of time, provided such a functional relationship exists.

* Numbers in the margin indicate pagination in the foreign text.

c. The hypothesis that the fluxes represent a Poisson process must be tested.

All problems are interrelated. For example, to solve problem b, problem a must be solved using the minimum number of cycles (if possible, one cycle). Henceforth, m_0 will denote the number of cycles on the basis of which one set of parameters is determined and the corresponding time (base time) will be denoted by T_0. /4

This article compares several methods for the estimation of the parameters (and thus also the solution of the first and, partially, the second formulated problems). The comparison is based both on theoretical concepts and also on the results obtained from applying these methods to trial fluxes obtained using Monte Carlo methods. Since for the trial fluxes the parameters are specified in advance, for a sufficiently large number of realizations we obtain the variances of the estimators given by the different methods. The analysis presented in the article can be used to select the most suitable method for the treatment of the results of actual physical measurements.

Section 2. The Case of Sufficiently Dense Fluxes

For sufficiently dense fluxes, two or more time instants will lie in some intervals t_0. In this case, the λ_i can be estimated as follows:

When the instants

$$t_1, t_2, \ldots, t_{k_i} \qquad (k_i \geq 2)$$

lie in the interval t_0,

$$\lambda_i = \frac{8(k_i - 1)}{t_{k_i} - t_1}. \tag{2.1}$$

When this holds for every λ_i in one cycle $4t_o$, this cycle will be the base time. Otherwise, the base time $T_o = 4mt_o$ can be selected so that this event will occur at least once for every λ_i.

Therefore, it is useful to change the form of formula (2.1). Henceforth, $k_i - 1$ will denote the total number of "steps," i.e. time intervals $\Delta t_{i\nu}$ between two successive time instants, included in the time interval t_o during which the i-th filter is active, counted in the base interval T_o. Then formula (2.1) can be rewritten in the form /5

$$\lambda_i = \frac{8(k_i-1)}{\sum_{\nu=1}^{k_i-1} \Delta t_{i\nu}} . \tag{2.2}$$

Here (and throughout the article), the subscript i denotes the number of the flux.

Let us estimate the accuracy of this method, which we will henceforth call the first approximation method.

It is known that the time between two successive points in a flux representing a Poisson process with parameter λ_i (henceforth we will assume that λ_i does not change in the base interval T_o) has an exponential distribution with expected value $1/\lambda_i$ and variance $1/\lambda_i^2$. Since the individual time intervals are statistically independent, for the sum of the intervals we have $s_i = 8(k_i - 1)$, which we denote by ξ_{s_i} (to emphasize that it is a random variable).

$$^1 M\xi_{s_i} = \frac{8(k_i-1)}{\lambda_i} ; \quad ^1 D\xi_{s_i} = \frac{8(k_i-1)}{\lambda_i^2} . \tag{2.3}$$

[1] [Throughout the article, M denotes expected value (E) and D denotes variance (Var).]

According to A.M. Lyapunov's central limit theorem, for sufficiently large s_i, ξ_{s_i} will be a normally distributed random variable. Therefore, we have (with probability 0.95) the approximate inequality

$$\left|\sum_{\nu=1}^{k_i-1}\Delta t_{i\nu} - \frac{8(k_i-1)}{\lambda_i}\right| < 1{,}96\sqrt{\frac{8(k_i-1)}{\lambda_i^2}}$$

or, after substitution and rearrangement,

$$\left|\lambda_i - \frac{8(k_i-1)}{\sum_{\nu=1}^{k_i-1}\Delta t_{i\nu}}\right| < 1{,}96\,\frac{\sqrt{8(k_i-1)}}{\sum_{\nu=1}^{k_i-1}\Delta t_{i\nu}}\,. \tag{2.4}$$

Denoting as usual the estimator of λ_i by $\tilde{\lambda}_i$, we obtain from /6
(2.4) for the relative error with probability 0.95

$$\left|\frac{\lambda_i-\tilde{\lambda}_i}{\tilde{\lambda}_i}\right| < \frac{1{,}96}{\sqrt{8(k_i-1)}}\,. \tag{2.5}$$

Thus, for a 10% error we must take $k_i = 49$. We note that for this value of k_i, it is legitimate to apply the A.M. Lyapunov theorem (central limit theorem).

Let us define the concept of a "sufficiently dense" flux more accurately. Suppose the base includes m_0 cycles. Then the total operating time of the i-th filter is $m_0 t_0$. When we have $k_i - 1$ "steps," the ratio

$$\ell_i = \frac{\sum_{\nu=1}^{k_i-1}\Delta t_{i\nu}}{m_0 t_0} \tag{2.6}$$

characterizes the density of the flux λ_i relative to the characteristic time t_0. Clearly $0 \le \ell_i \le 1$ and $\lambda_i \to \infty$ as $\ell_i \to 1$. Henceforth we will call ℓ_i the density coefficient. For $M\ell_i > 0.5$, we can consider the flux to be sufficiently dense (since ℓ_i is a random variable, and theoretically it is more convenient to consider its expected value $M\ell_i$).

Determination of Time Varying Flux. Suppose we have

$$\lambda_i = \lambda_i(t). \tag{2.7}$$

Let us consider two neighboring base intervals. The distance between their centers is T_0. To derive the relation (2.7), T_0 must be sufficiently small, so that the function (2.7) is linear on the interval T_0 (otherwise the estimators presented above will be biased). Then approximately

$$\left.\begin{array}{c} \lambda_i = a + bt; \\ \dfrac{\lambda_i(t+T_0)-\lambda_i(t)}{\lambda_i(t)} = \dfrac{bT_0}{a+bt} = \dfrac{bT_0}{\lambda_i(t)}. \end{array}\right\} \tag{2.8}$$

On the other hand, we have seen that for the given T_0 the error with which $\lambda_i(t)$ is determined can be, with probability 0.95, /7

$$1.96/\sqrt{8(k_i - 1)}.$$

Using the density coefficient ℓ_i, the last expression can be written in the form

$$\frac{1{,}96}{\sqrt{8(k_i-1)}} = \frac{1{,}96}{\sqrt{\lambda_i \sum_{\nu=1} \Delta t_{i\nu}}} = \frac{1{,}96}{\sqrt{\lambda_i \ell_i m_0 t_0}} = \frac{1{,}96}{\sqrt{\frac{1}{4}\lambda_i \ell_i T_0}} = \frac{3{,}92}{\sqrt{\lambda_i \ell_i T_0}}. \tag{2.9}$$

Comparing (2.8) and (2.9), we obtain the inequality

$$\frac{bT_0}{\lambda_i(t)} > \frac{3{,}92}{\sqrt{\lambda_i \ell_i T_0}}$$

or, after squaring both sides, multiplying by $\lambda_i T_0$ and rounding to $3.92 \approx 4$,

$$\frac{\beta^2 T_0^3}{\lambda_i(t)} > \frac{16}{\ell_i} \qquad (2.10)$$

It can be seen from formula (3.30) that for large $z = \lambda t_0$, (2.10) can be transformed as follows

$$T_0 \frac{\beta^2 T_0^2}{\lambda^2} > 16 \frac{1}{\lambda M\ell}$$

or

$$4 m t_0 \left(\frac{\Delta\lambda}{\lambda}\right)^2 > \frac{16}{\lambda\left(1 - \frac{9}{\lambda t_0}\right)} = \frac{16 t_0}{\lambda t_0 - 9}$$

i.e.

$$m > \frac{4}{\lambda t_0 - 9}\left(\frac{\lambda}{\Delta\lambda}\right)^2. \qquad (2.11)$$

Thus, for $\lambda t_0 = 29$ and $\Delta\lambda/\lambda = 0.1$, $m \geqslant 20$ cycles are required. Hence, the determination of time varying fluxes is a difficult problem. To improve its solution we must select better methods for estimating the density which will be discussed below.

Method of Successive Approximations

One can attempt to improve the accuracy of the estimates obtained above by considering them as first approximation estimates, denoted respectively by $\lambda_i^{(1)}$. For each interval Δt between two neighboring instants including two or more intervals t_{0i}, we write /8

$$\Delta t = \Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 \qquad (2.11)\ [sic]$$

where Δt_i is the part of Δt referring to t_{oi} (i.e. the active range of the i-th filter). Next, we partition

$$\Delta K_i = \frac{\lambda_i^{(1)} \Delta t_i}{\sum_{j=1}^{4} \lambda_j^{(1)} \Delta t_j} \tag{2.12}$$

and form the next approximation

$$\lambda_i^{(2)} = \frac{8(K_i - 1) + 8\sum \Delta K_i}{\sum_{\nu=1}^{m_{io}} \Delta t_{i\nu} + \sum \Delta t_i} = \frac{8(K_i - 1) + 8\sum \Delta K_i}{m_o t_o} \tag{2.13}$$

The accuracy can be further improved by replacing in (2.12) $\lambda_i^{(1)}$ by $\lambda_i^{(2)}$, etc. The accuracy of this method can be estimated from trial fluxes.

Section 3. General Discussion of the Problem

Suppose that the counter begins to operate at the time $t = 0$. In the general case, it records some time varying Poisson flux of pulses characterized by a piecewise continuous density function $\lambda(t)$, with $\lambda(t) = \lambda_i(t)$ for the i-th connected filter. When each flux is simple, $\lambda(t)$ will be a step function with period $4t_o$. Below, we will use the notation

$$Z(t) = \int_0^t \lambda(t)\,dt. \tag{3.1}$$

In particular, in the case of simplest fluxes, at the left endpoint of the interval t_{oi} preceded by m full cycles,

$$Z_{mi}(t) = m\sum_{j=1}^{4} \lambda_j t_o + \sum_{j=1}^{i-1} \lambda_j t_o. \tag{3.2}$$

In the general case [1], the probability that exactly k pulses /9
will appear in time t is

$$p_\kappa = e^{-z(t)}\frac{[z(t)]^\kappa}{\kappa!}. \tag{3.3}$$

Let us denote by p_μ ($\mu = 0,1,\ldots,7$) the probability of the event that the number of pulses can be represented in the form

$$\kappa = 8d + \mu. \tag{3.4}$$

Then we have the following formula:

$$p_\mu = \sum_{d=0}^{\infty} e^{-z(t)}\frac{[z(t)]^{8d+\mu}}{(8d+\mu)!}. \tag{3.5}$$

Below, we omit the argument t in z, and, conversely, consider p_μ as a function of z, i.e. $p_\mu(z)$. z is assumed to be a complex variable. Then, the functions $e^z p_\mu(z) = u_\mu(z)$ have the following properties:

$$\frac{d^8 u_\mu(z)}{dz^8} = u_\mu(z); \tag{3.6}$$

$$\frac{d^\kappa u_\mu(0)}{dz^\kappa} = \begin{cases} 1 & \text{for } \kappa = \mu; \\ 0 & \text{for } \kappa = 0,\ldots,\mu-1,\mu+1,\ldots,7. \end{cases} \tag{3.7}$$

From equations (3.6) with initial conditions (3.7) we obtain explicit expressions for the $u_\mu(z)$

$$u_0(z) = \frac{1}{8}\left[e^z + e^{\omega z} + \cdots + e^{\omega^7 z}\right] = \sum_{d=0}^{\infty}\frac{z^{8d}}{(8d)!}, \quad \omega = e^{\frac{\pi i}{4}}, \tag{3.8}$$

$$u_\mu(z) = \frac{d^{8-\mu}u_0(z)}{dz^{8-\mu}} = \frac{1}{8}\left[e^z + \omega^{8-\mu}e^{\omega z} + \cdots + \omega^{7(8-\mu)}e^{\omega^7 z}\right]. \tag{3.9}$$

Some properties of the functions $u_\mu(z)$ are discussed in [2,3].

We will prove the following lemma.

Lemma I. Let z be real and $z \to \infty$. Then

$$\lim_{z\to\infty} p_\mu(z) = \frac{1}{8}. \tag{3.10}$$

Proof. (3.9) implies /10

$$
\left.\begin{aligned}
&\left|\bar{p}_\mu(z)-\frac{1}{8}\right| = \frac{1}{8}\left|\omega^{8-\mu}e^{(\omega-1)z}+\dots+\omega^{7(8-\mu)}e^{(\omega^7-1)z}\right| \leq \\
&\leq \frac{1}{8}\sum_{j=1}^{7}\left|\omega^{j(8-\mu)}e^{(\omega^j-1)z}\right| = \frac{1}{8}\sum_{j=1}^{7}\left|e^{(\omega^j-1)z}\right| = \frac{1}{8}\sum_{j=1}^{7}e^{\mathrm{Re}\left[(\omega^j-1)z\right]} = \\
&= \frac{1}{8}\sum_{j=1}^{7}e^{\mathrm{Re}(\omega^j-1)\cdot z} = \frac{1}{8}\sum_{j=1}^{7}e^{\left(\cos\frac{\pi j}{4}-1\right)z} < \\
&< \frac{7}{8}e^{-\left(1-\frac{\sqrt{2}}{2}\right)z},
\end{aligned}\right\} \tag{3.11}
$$

which proves the lemma.

Using estimator (4.1), we obtain, for example, the result that the equality $p_\mu(z) = 1/8$ is valid starting with $z = 23$ with a relative error smaller than 1%. It can be seen from Table I that the relative error does not exceed 1% for all practical purposes already at $z = 18$. Thus (4.1) is a sufficiently good estimator.

We note that lemma 1 can also be derived from the general results for Markov processes; however, the estimator for the accuracy is not as good as (4.1).

Let us return to formula (3.2). We have

$$
p_\mu = p_\mu(z_{mi}(t)). \tag{3.12}
$$

We will consider the interval t_{oi}.

The probability that exactly k instants will lie in the interval t_{oi} can be expressed as follows:

$$
\left.\begin{aligned}
&\text{for } k = 0 \qquad \tilde{P}_0 = \sum_{\mu=0}^{7} p_\mu \sum_{j=0}^{7-\mu} e^{-\lambda_i t_0}\frac{(\lambda_i t_0)^j}{j!}; \\
&\text{for } k > 0 \qquad \bar{P}_k = \sum_{\mu=0}^{7} p_\mu \sum_{j=0}^{7} e^{-\lambda_i t_0}\frac{(\lambda_i t_0)^{8(k-1)+j+8-\mu}}{[8(k-1)+j+8-\mu]!}.
\end{aligned}\right\} \tag{3.13}
$$

TABLE 1. $p_\mu(z)$ /11

z \ μ	0	I	2	3	4	5	6	7
0,I	0.90484	0.09048	0.00452	0.000I5	0	0	0	0
0.2	0.8I873	0.I6375	0.0I637	0,00I09	0,00005	0	0	0
0,3	0.740[illegible]2	0.22225	0.03334	0.00333	0,00025	0,00002	0	0
0.4	0.67032	0,268I3	0,05363	0.[illegible]7I5	0.00072	0,00006	0	0
0.5	0.60653	0,30327	0,07582	0.0I264	0.00I58	0.000I6	0.0000I	0
0.6	0.5488I	0.32929	0,09879	0,0I976	0,00296	0.00036	0.00004	0
0.7	0,49659	0,3476I	0.I2I66	0.02839	0.00497	0.00070	0.00008	0,0000I
0.8	0,44933	0.35946	0,I4379	0.03834	0,00767	0.00I23	0.000I6	0.00002
0.9	0,40657	0,3659I	0.I6466	0.04940	0.0IIII	0,00200	0.00030	0.00004
I,0	0,36789	0.36788	0.I8394	0.06I3I	0.0I533	0.00307	0.0005I	0.00007
I,2	0.30I23	0.36I44	0.2I686	0.08674	0.02602	0.00625	0.00I25	0,0002I
I.4	0.24669	0,34525	0,24I67	0,II278	0.03947	0.0II05	0.00258	0,00052
I,6	0,202II	0,32307	0,25[illegible]43	0.I3783	0.055I3	0.0I764	0.00470	0.00I08
I,8	0,I657[illegible]	0,29763	0,26780	0,I6067	0,07230	0.02603	0.0078I	0.0020I
2,0	0,I36I9	0,27086	0.2707I	0.I8045	0,09[illegible]22	0,036[illegible]9	0.0I203	0.00344
3.0	0.05789	0.I5206	0,22485	0,22426	0.I6809	0.I0083	0,0504I	0,02I60
4,0	0.04809	0.08649	0.I5I82	0.I9729	0,I960I	0.I5649	0,I0425	0.05956
5.0	0.07207	0.06997	0,I0236	0,I[illegible]362	0.I7890	0.I7679	0,I4669	0,I0460
6.0	0.I0607	0,08383	0,08596	0,III78	0,I45I2	0,I65[illegible]2	0,I6285	0,I3857
7.0	0.I3274	0.I0838	0,09356	0,09739	0,II76I	0,I4I92	0,I56I0	0,I523I
8,0	0,I4444	0,I2889	0,II094	0,I0I2I	0,I0554	0,I2I28	0,I3908	0,I4862
9.0	0I4282	0.I[illegible]66	0,I2647	0,II339	0,I07I2	0,III37	0,I2358	0,I3659
I0,0	0.I3442	0,I3836	0,I3[illegible]48	0.I2504	0,II556	0,III63	0,II554	0.I2497
II,0	0,I2576	0,I3256	0,I3494	0,I3I50	0,I24[illegible]4	0,II743	0,II5[illegible]6	0,II85I
I2,0	0.I206I	0,I26I4	0,I3I00	0,I3235	0,I2939	0,I2386	0.II899	0,II765
I3,0	0.II960	0.I2208	0,I2628	0,I2972	0.I3040	0,I2792	0,I2372	0,I2028
I4.0	0,I2I32	0.I2I06	0,I23II	0,I2627	0,I2868	0,I2894	0,I2689	0,I2373
I5,0	0,I2383	0,I22I5	0,I22I4	0,I238I	0,I26I7	0,I2785	0.I2786	0,I26I9
I6,0	0,I2572	0,I2396	0,I228I	0 I2294	0,I2428	0,I2604	0.I27I9	0,I2706
I7,0	0,I2647	0.I254I	0.I24II	0.I2333	0,I2353	0,I2459	0,I2589	0,I2667
I8,0	0.I2627	0.I2604	0,I252I	0.I2425	0,I2373	0,I2396	0,I2479	0,I2575
I9,0	0,I2562	0,I2595	0,I2573	0.I2508	0,I2438	0,I2405	0,I2427	0I2492
20,0	0I2500	0,I2550	0,I257I	0,I255I	0,I2500	0,I2450	0,I2429	0,I2449
30,0	0,I2497	0,I2500	0,I2503	0 I2504	0,I25[illegible]3	0,I2500	0,I2497	0,I2496

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/13

These formulas are exact when we use the expressions for /14
p_μ derived above, and in the case of time varying fluxes replace $\lambda_1 t_0$ by

$$\int_0^{t_0} \lambda(t)\,dt.$$

Using (3.13), we can write down an exact formula for the expected number M_k of instant k in the interval t_0:

$$M_K = \sum_{K=1}^{\infty} \bar{p}_K K = \sum_{\mu=0}^{7} p_\mu \sum_{j=0}^{7} e^{-\lambda_i t_0} \sum_{K=1}^{\infty} K \frac{(\lambda_i t_0)^{8(K-1)+j+8-\mu}}{[8(K-1)+j+8-\mu]!} \tag{3.14}$$

We introduce the notation

$$m_\mu(z) = \sum_{K=0}^{\infty} (K+1) e^{-z} \frac{z^{8K+\mu}}{(8K+\mu)!} \qquad (\mu=0,1,\ldots,7). \tag{3.15}$$

The identity

$$m_\mu(z) = \frac{z}{8} p_{\mu-1}(z) + \left(1-\frac{\mu}{8}\right) p_\mu(z). \tag{3.16}$$

is easily verified.

For $\mu = 0$, in this identity we replace $\mu - 1$ by 7. Using (3.16) and Table 1, we can calculate $m_\mu(z)$ for a set of values of z.

For $\mu = 8 + \nu$ ($\nu = 0,1,\ldots,7$), we obtain

$$\sum_{K=0}^{\infty} (K+1) e^{-z} \frac{z^{8K+\mu}}{(8K+\mu)!} = m_\nu(z) - p_\nu(z). \tag{3.17}$$

Now, formula (3.14) can be written in the following form:

$$M_K = \sum_{\mu=0}^{7} p_\mu \left\{ \sum_{j=0}^{\mu-1} m_{(8+j-\mu)}(\lambda_i t_o) + \sum_{j=\mu}^{7} \left[m_{(j-\mu)}(\lambda_i t_o) - p_{(j-\mu)}(\lambda_i t_o) \right] \right\}. \tag{3.18}$$

When $\mu = 0$, the formula does not include the first sum in the braces.

M_k can always be calculated from formula (3.18) involving a finite number of terms. The expression p_μ outside the braces /16 depends on the argument in (3.2).

Under the conditions of Lemma I, i.e., when we can set $p_\mu = 1/8$, the following holds

Lemma 2.

$$\lim_{p_\mu \to \frac{1}{8}} M_K = \frac{\lambda_i t_o}{8}. \tag{3.19}$$

Proof. For compactness, we write $\lambda_i t_o = z$.

$$\sum_{\mu=0}^{7} \frac{1}{8} \left\{ \sum_{j=0}^{\mu-1} m_{(8+j-\mu)}(z) + \sum_{j=\mu}^{7} \left[m_{(j-\mu)}(z) - p_{(j-\mu)}(z) \right] \right\} =$$

$$= \frac{1}{8} e^{-z} \sum_{K=0}^{\infty} \left\{ \sum_{\mu=0}^{7} \sum_{j=0}^{\mu-1} (K+1) \frac{z^{8K+8+j-\mu}}{(8K+8+j-\mu)!} + \sum_{\mu=0}^{7} \sum_{j=\mu}^{7} \left[(K+1) \frac{z^{8K+j-\mu}}{(8K+j-\mu)!} - \frac{z^{8K+j-\mu}}{(8K+j-\mu)!} \right] \right\} = \frac{1}{8} e^{-z} \sum_{\chi=0}^{\infty} A_\chi z^\chi. \tag{3.20}$$

Let us find the expression for A_χ in (3.20). Let $\chi = 8k + \chi_1$:

$$A_\chi = \sum_{\mu=0}^{7} \sum_{j=0}^{\mu-1} \frac{K+1}{(8K+\chi_1)!} + \sum_{\mu=0}^{7} \sum_{j=\mu}^{7} \frac{K}{(8K+\chi_1)!} = \sum_{\mu=0}^{7} \left[\frac{(K+1)\psi_1(\mu) + K\psi_2(\mu)}{(8K+\chi_1)!} \right]$$

where $\psi_1(\mu)$ and $\psi_2(\mu)$ are the number of terms in the inner sums satisfying the conditions at μ, j and χ_1. For example, we have $\psi_1(0) = 0$ (first sum vanishes), $\psi_2(0) = 1$ for any χ_1 (then we take $j = \chi_1$, which is always possible since $0 \leq \chi_1 \leq 7$). For $\mu > 0$, $\psi_1(\mu) = 1$ when we can take $j = \chi_1 + \mu - 8$, obtaining $0 \leq j \leq \mu - 1$, i.e. when $\mu \geq 8 - \chi_1$. This implies $\sum_\mu \psi(\mu) = \chi_1$. Next $\psi_2(\mu) = 1$ if $j = \mu + \chi_1 \leq 7$ exists, i.e. $\mu \leq 7 - \chi_1$. Hence $\sum_\mu \psi_2(\mu) = 8 - \chi_1$. Thus, for $\chi \geq 1$,

$$A_\chi = \frac{(k+1)\chi_1 + k(8-\chi_1)}{(8k+\chi_1)!} = \frac{8k+\chi_1}{(8k+\chi_1)!} = \frac{1}{(8k+\chi_1-1)!} = \frac{1}{(\chi-1)!}$$

and for $\chi = 0$, i.e. $\chi_1 = 0$ and $k = 0$, $A_\chi = 0$. /17

Now (3.20) gives

$$\frac{1}{8} e^{-z} \sum_{\chi=0}^{\infty} A_\chi z^\chi = \frac{1}{8} e^{-z} \sum_{\chi=1}^{\infty} \frac{z^\chi}{(\chi-1)!} = \frac{z}{8},$$

which proves the statement of the lemma.

Similarly, as in the derivation of lemma 2, we obtain for the variance D_k as $p_\mu \to 1/8$ the expression

$$D_k = \frac{1}{8} e^{-z} \sum_{k=0}^{\infty} \sum_{\nu=0}^{7} (8k^2 + 2k\nu + \nu) \frac{z^{8k+\nu}}{(8k+\nu)!} - (M_k)^2, \qquad (3.21)$$

which can be expressed in a different form.

We introduce the notation

$$d_\mu(z) = \sum_{k=0}^{\infty} (k+1)^2 e^{-z} \frac{z^{8k+\mu}}{(8k+\mu)!}; \quad (\mu = 0, 1, \ldots, 7) \qquad (3.22)$$

Similarly as in (3.16) we obtain

$$d_\mu(z) = \frac{z}{8} m_{\mu-1}(z) + \left(1 - \frac{\mu}{8}\right) m_\mu(z). \qquad (3.23)$$

Now we can write $D_k + (M_k)^2$ in the form:

$$
\left.
\begin{aligned}
D_k + (M_k)^2 &= \frac{1}{8}\sum_{\nu=0}^{7}\left[8d_\nu(z) - (16-2\nu)m_\nu(z) + (8-\nu)p_\nu(z)\right] = \\
&= \frac{1}{64}\sum_{\nu=0}^{7}\left[z^2 p_{\nu-2}(z) + z p_{\nu-1}(z) + \nu(8-\nu)p_\nu(z)\right].
\end{aligned}
\right\} \tag{3.24}
$$

We note that for $\nu < 2$, $p_{\nu-2}$ must be replaced by $p_{\nu+6}$. Taking into consideration that $M_k = z/8$ and $\sum_{\nu=0}^{7} p_\nu(z) = 1$, we finally obtain

Lemma 3

$$
\lim_{p_\mu \to \frac{1}{8}} D_k = \frac{1}{64}\left[z + \sum_{\nu=0}^{7}\nu(8-\nu)p_\nu(z)\right]; \quad z = \lambda_i t_0. \tag{3.25}
$$

For small z, the value of D_k can be calculated with the aid of Table 1. For $z \leq 20$, the values of $64D_k$ are given in Table 2. For $z > 20$, we can use with sufficient accuracy $p_\nu(z) = 1/8$. We then have /18

$$
D_k = \frac{1}{64}(\lambda_i t_0 + 10{,}5) \tag{3.26}
$$

Finally, let us estimate the density coefficient ℓ_i which was mentioned above. We drop the subscript i. Then we have the following exact formula

$$
\left.
\begin{aligned}
M\ell &= \frac{1}{t_0}\sum_{\alpha=0}^{7}p_\alpha\int_0^{t_0} e^{-\lambda t}\frac{(\lambda t)^{7-\alpha}}{(7-\alpha)!}\left[\int_0^{t_0-t} x\sum_{k=1}^{\infty}e^{-\lambda x}\frac{(\lambda x)^{8k-1}}{(8k-1)!}\cdot\varphi(x)\lambda\,dx\right]\lambda\,dt, \\
\varphi(x) &= \sum_{\beta=0}^{7}e^{-\lambda(t_0-t-x)}\frac{[\lambda(t_0-t-x)]^\beta}{\beta!}.
\end{aligned}
\right\} \tag{3.27}
$$

where

The interpretation of the formula is the successive summing (integration) of arbitrary values ℓ with the conditions:

TABLE 2

z	Mℓ	$64 D_K$	z	Mℓ
0	0	0	22	0.59095
I	0.00000	6,997	24	0.62501
2	0.00003	II.6I7	26	0.65385
3	0.00050	I5.084	28	0.67857
4	0,00307	I6,538	30	0,70000
5	0.0I080	I6,954	40	0,77500
6	0,02688	I7,025	50	0.82000
7	0,05297	I7.29I	60	0.85000
8	0,08865	I7.969	70	0.87I43
9	0.I3I76	I9.UI3	80	0.88750
I0	0,I7932	20.243	90	0.90[illegible]0
II	0,22843	21.479	I00	0,91[illegible]00
I2	0,27677	22,620		
I3	0,32273	23 648		
I4	0,36542	24,600		
I5	0,40447	25.532		
I6	0,43987	26,480		
I7	0,47I82	27,660		
I8	0.500[illegible]3	28,465		
I9	0,52664	29,483		
20	0,550I6	30,500		

a) 7 - α points of the flux precede the first instant in the interval t_o (the probability of this event is p_α).

b) The first instant coincides with the instant t.

c) The sum of the lengths of the "steps" is x.

d) The number of instants is k + 1 (k "steps").

e) There are β remaining points in the interval t_o after the "steps."

The integral in the square brackets is transformed to the form

$$e^{-\lambda(t_0-t)}\sum_{K=1}^{\infty}\sum_{\beta=0}^{7}\int_0^{t_0-t}\frac{(\lambda x)^{8K-1}}{(8K-1)!}\cdot\frac{[\lambda(t_0-t-x)]^{\beta}}{\beta!}\lambda x\,dx = e^{-\lambda(t_0-t)}\frac{8}{\lambda}\sum_{K=1}^{\infty}K\sum_{\beta=0}^{7}\frac{[\lambda(t_0-t)]^{8K+\beta+1}}{(8K+\beta+1)!}$$

Substituting in (3.27) and integrating, we obtain

$$M\ell = \frac{8e^{-\lambda t_0}}{\lambda t_0}\sum_{\alpha=0}^{7}p_\alpha\sum_{K=1}^{\infty}K\sum_{\beta=0}^{7}\frac{(\lambda t_0)^{8K+\beta+9-\alpha}}{(8K+\beta+9-\alpha)!}. \tag{3.28}$$

We let $\lambda t_0 = z$, assume $p_\infty \to 1/8$ and write /19

$$\varphi_\alpha(z) = \sum_{K=0}^{\alpha}e^{-z}\frac{z^K}{K!}$$

for the Poisson distribution. Then, transforming the series (3.28) analogously as in the derivation of lemma 2, we obtain

Lemma 4

$$\lim_{p_\alpha\to\frac{1}{8}} M\ell = \left(1-\frac{9}{z}\right)\left(1-\varphi_7(z)\right)+\frac{9}{z}e^{-z}\frac{z^8}{8!}. \tag{3.29}$$

Table 2 gives the values of the right member in (3.29) for some z. It can be seen from Table 2 that starting with z = 15,

$$M\ell \approx 1-\frac{9}{z}. \tag{3.30}$$

with an accuracy up to 1%.

Section 4. Second Approximation Method

We will consider the base interval $T_0 = 4m_0t_0$. When the left endpoint of this interval is sufficiently far from the point at which the counter begins to work, the expected number of instants included in the intervals in which the i-th filter operates is, according to lemma 2,

$$\frac{1}{8}\lambda_i m_0 t_0.$$

Suppose that the actual number of instants is n_i. Then

$$\lambda_i^{(1)} = \frac{8n_i}{m_0 t_0} \tag{4.1}$$

can be taken as the first approximation. The accuracy can be further improved along the lines described in Section 2. This method of successive approximations can also be applied to low density fluxes. The accuracy of the method can be estimated on the basis of experimental trial problems.

For sufficiently dense fluxes, when both first approximation methods can be applied, their accuracy can be compared theoretically. In fact, taking into account (3.26), the variance for (4.1) is

$$D_2 = \frac{m_0}{64}(\lambda_i t_0 + 10{,}5)\frac{64}{(m_0 t_0)^2} = \frac{\lambda_i t_0 + 10{,}5}{m_0 t_0^2}. \tag{4.2}$$ /20

Transforming (2.4), we have for the first method

$$D_1 = \frac{8(k_i - 1)}{\left(\sum_{\nu=1}^{k_i-1} \Delta t_{i\nu}\right)^2}. \tag{4.3}$$

Expression (4.3) is not convenient for a comparison, since it involves only random variables. However, passing to expected values, we can write

$$\sum_{\nu=1}^{K_i-1} \Delta t_{i\nu} = \frac{8(K_i-1)}{\lambda_i},$$

i.e.

$$D_1 \approx \frac{\lambda_i^2}{8(K_i-1)} = \frac{\lambda_i}{\sum_{\nu=1}^{K_i-1} \Delta t_{i\nu}} = \frac{\lambda_i t_0}{m_0 t_0^2 \ell_i}. \tag{4.4}$$

Using formula (3.29) for $M\ell$, we obtain

$$D_1 \approx \frac{z}{m_0 t_0^2 \left[\left(1-\frac{9}{z}\right)\left(1-\varphi_7(z)\right) + 9e^{-z}\frac{z^7}{8!}\right]}. \tag{4.5}$$

Two conclusions follow from the above. First, for small z, D_1 increases enormously, which was to be expected. In fact, expanding $M\ell$ in a series in powers of z we obtain

$$M\ell = \frac{z^9}{10!} - \frac{9z^{10}}{11!} + \cdots, \tag{4.6}$$

i.e.

$$D_1 \sim \frac{1}{z^8}. \tag{4.7}$$

as $z \to 0$.

For a sufficiently high density, we can use formula (3.30) for which we obtain

$$D_1 \approx \frac{z}{m_0 t_0^2 \left(1-\frac{9}{z}\right)}. \tag{4.8}$$

The comparison with (4.2) gives

$$\frac{D_2}{D_1} = \left(1 + \frac{10.5}{z}\right)\left(1 - \frac{9}{z}\right), \quad (4.9)$$ /21

i.e., $D_2/D_1 \to 1$ as $z \to \infty$. (5.0)

Thus, for high densities, the first method may be more efficient, since

$$\left(1 + \frac{10.5}{z}\right)\left(1 - \frac{9}{z}\right) = 1 + \frac{1.5}{z} - \frac{94.5}{z^2}, \quad (5.1)$$

which is greater than 1 starting with $z = 63$. However, this advantage is very small and it approaches zero as z approaches infinity. D_2/D_1 attains its largest value, 1.006, at $z = 126$.

We note that using formula (4.5) for D_1 and formula (3.25) for D_2 (instead of (3.26)), we can calculate more accurately the values of D_2/D_1 for a set of values of z on the basis of Tables 1 and 2.

The corresponding values are given in Table 3.

Section 5. Method of Least Squares

The method of least squares enables us to treat the base interval without using lemmas 1 and 2. The successive intervals between the instants included in the base interval whose length is $4m_o t_o$ are taken as the base. Suppose that altogether [?] instants are included in the interval, i.e. [?] intervals. Let j denote the number of the interval. For a fixed j-th interval, each i-th filter operates on it during n_{ji} ($n_{ji} = 0,1,\ldots$) time intervals $\Delta t_{i\nu j}$ ($\nu = 1,\ldots,n_{ji}$). Thus, the expected number of pulses in the j-th interval is

$$M\ell_j = \sum_{i=1}^{4} \lambda_i \sum_{\nu=1}^{n_{ji}} \Delta t_{i\nu j} \quad (5.1) \text{ [sic]}$$

TABLE 3 /22

z	$\frac{D_2}{D_1}$
3	0,002
4	0,013
5	0,037
6	0,076
7	0,131
8	0,1[illegible]9
9	0,278
10	0,363
11	0,446
12	0,522
13	0.587
14	0,642
15	0,688
16	0.728
17	0,768
18	0,791
19	0,817
20	0,839
30	0,945

where the inner sum is zero when $n_{ji} = 0$. We note that the variance of the number of pulses mentioned is also given by expression (5.1), i.e.: /23

$$D\ell_j = M\ell_j . \tag{5.2}$$

Since in fact $\ell_j = 8$, formally applying the method of least squares, we obtain

$$F(\lambda) = \sum_{j=1}^{N} \frac{(M\ell_j - \ell_j)^2}{D\ell_j} = \min . \tag{5.3}$$

for estimating the $\lambda = (\lambda_1, \ldots, \lambda_4)$. The resulting problem is nonlinear. However, taking into account (5.2) and the fact that $M\ell_j = \ell_j = 8$, we note that the variances $D\ell_j$ are approximately equal. Hence, they can be omitted and the problem takes on the form:

$$F_1(\lambda) = \sum_{j=1}^{N} \left(\sum_{i=1}^{4} \lambda_i \sum_{\nu=1}^{n_{ij}} \Delta t_{i\nu j} - 8 \right)^2 \Rightarrow \min . \tag{5.4}$$

From the above, we obtain in the usual manner for the λ_i the system of equations

$$\sum_{i=1}^{4} A_{i\kappa} \lambda_i = A_{0\kappa} \quad (\kappa = 1, \ldots, 4), \tag{5.5}$$

where

$$\left. \begin{aligned} A_{0\kappa} &= 8 \sum_{j=1}^{N} \sum_{\nu=1}^{n_{ji}} \Delta t_{\kappa\nu j} ; \\ A_{i\kappa} &= \sum_{j=1}^{N} \left(\sum_{\nu=1}^{n_{ji}} \Delta t_{i\nu j} \cdot \sum_{\nu=1}^{n_{j\kappa}} \Delta t_{\kappa\nu j} \right) . \end{aligned} \right\} \tag{5.6}$$

We note that when this method is used, any interval, which is arbitrarily close to the point at which the counter begins to operate can be used as the base interval. However, it is assumed that the fluxes are simple in the base interval.

It can be seen from the results obtained on the basis of the method of least squares presented below in Section 6 that this method gives estimators with satisfactory variances but with considerable bias. To clarify the reason for this phenomenon, we will consider the estimation of the parameter λ of a single flux. Instead of (5.4), we obtain

$$F_1(\lambda) = \sum_{j=1}^{N} (\lambda \Delta t_j - 8)^2 = \min, \tag{5.7}$$ /21

i.e. for the estimator $\tilde{\lambda}$ we have

$$\tilde{\lambda} = \frac{8 \sum_{j=1}^{N} \Delta t_j}{\sum_{j=1}^{N} (\Delta t_j)^2} = \frac{8 \frac{1}{N} \sum_{j=1}^{N} \Delta t_j}{\frac{1}{N} \sum_{j=1}^{N} (\Delta t_j)^2}. \tag{5.8}$$

As N → ∞ the numerator and denominator converge in the probability sense to the corresponding expected values and we obtain

$$\lim_{N \to \infty} \tilde{\lambda} = \frac{\lambda}{2} \tag{5.9}$$

i.e., in the limit the bias in the estimate is half the estimated parameter. It can be seen from the table that the bias increases as N increases.

However, if we use

$$\left(\lambda \sum_{j=1}^{N} \Delta t_j - 8N\right)^2 = \min = 0 \tag{5.10}$$

for the estimation instead of (5.7), we obtain

$$\tilde{\lambda} = \frac{8}{\frac{1}{N} \sum_{j=1}^{N} \Delta t_j} \to \lambda \tag{5.11}$$

i.e. an unbiased estimator. This leads to the idea of improving the accuracy of the least squares method by summing in (5.4) terms with the same structure. Let us consider the j-th interval ΔT_j. It is broken up into intervals with different filter numbers i as follows:

$$\Delta T_j = \Delta t_{i_1, 1j} + \Delta t_{i_1+1, 1j} + \Delta t_{i_1+2, 1j} + \dots + \Delta t_{i_1, 2j} + \dots \tag{5.12}$$

We note that all terms in the sum (5.12) are equal to t_0 except the first and last term. We will assume that the structure of ΔT_j is determined by a pair of numbers $s = (i_j, n_j)$, where n_j is the total number of terms in the partition (5.12). /25

Let N_s denote the total number of intervals ΔT_j with structure s, and Ω_s the set j corresponding to the given s. Then

$$N = N_{1,1} + N_{2,1} + N_{3,1} + N_{4,1} + N_{1,2} + \cdots + N_{1,3} + \cdots \tag{5.13}$$

Now, instead of (5.3), we introduce the following sum of squares

$$\Phi(\lambda) = \sum_{s} \frac{1}{N_s}\Big(\sum_{j\in\Omega_s}\sum_{i=1}^{2}\lambda_i\sum_{\nu=1}^{n_{ij}}\Delta t_{i\nu j} - 8N_s\Big)^2. \tag{5.14}$$

The factors $1/N_s$ in front of the brackets were introduced to take into account the variances.

Because of the considerable difficulties connected with the minimization of (5.14), the problem can be solved approximately by summing only over certain structures s. The case when the sum includes only structures of the form s = (i,1) will be called the first approximation. It can be easily verified that this approximation coincides with the first approximation method described above. In the second approximation, structures of the from s = (i,2) are also included in the sum. For this case, (5.14) reduces to the form

$$\Phi_2(\lambda) = \sum_{i=1}^{4}\frac{1}{N_{i,1}}\Big(\lambda_i\sum_{\nu=1}^{N_{i,1}}\Delta t_{i\nu 1} - 8N_{i,1}\Big)^2 + \sum_{i=1}^{4}\frac{1}{N_{i,2}}\Big(\lambda_i\sum_{\nu=1}^{N_{i,2}}\Delta t''_{i\nu 2} + \lambda_{i+1}\sum_{\nu=1}^{N_{i,2}}\Delta t'_{i+1,\nu 2} - 8N_{i,2}\Big)^2 \tag{5.15}$$

(for i = 4 we take i + 1 = 1).

In formula (5.15), one prime in the Δt indicates that the interval Δt lies at the left endpoint of the corresponding interval t_o, and two primes indicate that it lies at the right endpoint.

The following notation is convenient: /26

$$\left.\begin{aligned} &\sum_{\nu=1}^{N_{i,1}} \Delta t_{i\nu_1} = s_{ii}; \\ &\sum_{\nu=1}^{N_{i,2}} \Delta t''_{i\nu_2} = s_{i,i+1}; \quad \sum_{\nu=1}^{N_{i,2}} \Delta t'_{i+1,\nu_2} = s_{i+1,i}. \end{aligned}\right\} \tag{5.16}$$

In more general form s_{ij} will denote the sum of the $\Delta t_{i\nu_j}$ over all intervals t_{oi} in the base and their subintervals Δt in the intervals between neighboring instants, with one endpoint in t_{oi} and the other endpoint in t_{oj}. Then formula (5.15) can be rewritten as follows:

$$\Phi_2(\lambda) = \sum_{i=1}^{k} \left\{ \frac{1}{N_{i,1}} (\lambda_i s_{ii} - 8N_{i,1})^2 + \frac{1}{N_{i,2}} (\lambda_i s_{i,i+1} + \lambda_{i+1} s_{i+1,i} - 8N_{i,2})^2 \right\}. \tag{5.17}$$

We again obtain the system of equations (5.5) for the λ_i, but instead of (5.6), the expressions for the coefficients are:

$$\left.\begin{aligned} &A_{0k} = 8(s_{k,k-1} + s'_{k,k} + s_{k,k+1}); \\ &A_{ii} = \frac{(s_{ii})^2}{N_{i,1}} + \frac{(s_{i,i+1})^2}{N_{i,2}} + \frac{(s_{i,i-1})^2}{N_{i-1,2}}; \\ &A_{i,i+1} = \frac{s_{i,i+1} \cdot s_{i+1,i}}{N_{i,2}}; \quad A_{i,i-1} = \frac{s_{i,i-1} \cdot s_{i-1,i}}{N_{i-1,2}}. \end{aligned}\right\} \tag{5.18}$$

For the more general case, the formulas become more cumbersome. The terms

$$\frac{1}{N_{i,3}} (\lambda_i s_{i,i+2} + \lambda_{i+1} \bar{s}_{i+1} + \lambda_{i+2} s_{i+2,i} - 8N_{i,3})^2 + \dots \tag{5.19}$$

must be added inside the braces in (5.17). In the above $\bar{s}_{i+1} = N_{is} t_o$, etc. Here, difficulties arise even in writing down the formulas, since for $i > 4$, it is replaced by the residual modulo four. Thus, for example, the expression $s_{i+2,i}$ in (5.19) becomes $s_{5,3} = s_{1,3}$ for $i = 3$, and it is not clear whether the left or right endpoints of the intervals t_{0i} are meant. Therefore, primes must be used as in formula (5.15). We note that such an ambiguity does not arise in (5.17), since, for example, for $i = 4$

$$s_{i+1,i} = s_{5,4} = s_{1,4}.$$

This can have only one meaning, namely, the right endpoint lies /27
at t_{01} and the left endpoint at t_{04}, since, if the converse were true, we would obtain $s = (i,4)$ instead of the structure $s = (i,2)$.

The results obtained from solving the problem in the second approximation by the method of least squares are presented in Section 6.

Section 6. Modeling of Trial Fluxes and Calculation of Statistical Characteristics of Different Methods

Each flux is modeled as follows: From time $t = 0$, the intervals between the pulses (which have an exponential distribution with parameter λ_i) are laid off on the axis, i.e. uniformly distributed random numbers ξ_k on the interval (0,1) are selected and

$$\Delta t_k = \frac{1}{\lambda_i} \ln(1-\xi_k). \tag{6.1}$$

is used. When a model of a time varying flux is required

$$\lambda_i = \lambda_i(t), \tag{6.2}$$

the value corresponding to

$$t_{k+1} = \sum_{j=1}^{k+1} \Delta t_j \tag{6.3}$$

is substituted in (6.1) for λ_i.

Next, the time axis is partitioned into subintervals of length t_0, in which the instants are entered as follows: When $8k + \nu_1$ points lie in the first interval, the points whose numbers are multiples of eight are taken as the instants and the number ν_1 is stored in memory. Next, we consider the second interval of length t_0. If it includes $8k_2 + \nu_2$ points from the second flux, for $8k_2 + \nu_2 \geq 8 - \nu_1$, the point with the number $8 - \nu_1$ is taken as the instant, etc. When $8k_2 + \nu_2 < 8 - \nu_1$ (i.e. $k_2 = 0$, $\nu_1 + \nu_2 < 8$), no instant lies in the interval under consideration and the number $\nu_1 + \nu_2$ is stored in memory, etc.

Thus a sequence of instants is generated which simulates the sequence transmitted by the physical counter. Each instant /28 is characterized by its own time t_j ($j = 1,\ldots,n$). In the study of the simplest fluxes, we used $n = 1000$. The trial fluxes were modeled for the following variants of the densities λ_i:

Variants	λ_1	λ_2	λ_3	λ_4
I	0.267	0.200	0.500	0.133
II	0.800	0.600	1.500	0.400
III	2.400	1.800	4.500	1.200

N (N = 50) realizations of the fluxes were calculated for each variant. From these, the same base intervals were selected for each variant: a) from $t = 0$ to $t = 4m_0t_0 = 4$ min ($m_0 = 5$, $t_0 = 12$ sec); b) from $t = 4$ min to $t = 8$ min; and c) from $t = 4$ min to $t = 20$ min. Variant b) is the basic variant. Variant a) was used to illustrate the effect of the transient regime and variant c) to illustrate the effect of the length of the base.

The values $\lambda_1^{(\nu)}$, $\lambda_2^{(\nu)}$, $\lambda_3^{(\nu)}$, $\lambda_4^{(\nu)}$ (ν denotes the number of the realization) and the first and second sample moments

$$\bar{\lambda}_i = \frac{1}{N}\sum_{\nu=1}^{N}\lambda_i^{(\nu)}; \tag{6.4}$$

$$\bar{\mu}_{i_1 i_2} = \frac{1}{N}\sum_{\nu=1}^{N}(\lambda_{i_1}^{(\nu)} - \bar{\lambda}_{i_1})(\lambda_{i_2}^{(\nu)} - \bar{\lambda}_{i_2}). \tag{6.5}$$

were calculated for a particular estimation method and the selected variants.

In addition the second sample moments about the true values λ_i were calculated

$$\tilde{\mu}_{i_1 i_2} = \bar{\mu}_{i_1 i_2} + (\bar{\lambda}_{i_1} - \lambda_{i_1})(\bar{\lambda}_{i_2} - \lambda_{i_2}). \tag{6.6}$$

The following principle was used to reduce the total number of variants. First, the methods were compared on the basis of variant II. Next, some variants were calculated, using particular /29
methods to illustrate the theoretical results discussed above, and to clarify relations not amenable to a theoretical study.

The results of the calculations are presented in Tables 4-7. The following conclusions can be drawn on the basis of the tables:

1. The sample correlation coefficients are negligibly small (Table 7). Therefore, the remaining tables give only the variances.

2. Table 6 shows the effect of the transient regime. For variant Ia, for λ_1 the error in $\bar{\lambda}_1$ is 30%; however, for IIb, the error already decreases considerably. Moreover, for Ia, for $i = 4$ the error is already negligible.

3. Table 5 shows that for λ_3 ($z = \lambda t_0 = 54$), the accuracy of the first approximation using the first method is roughly

TABLE 4. VALUES OF $\overline{\lambda}_i$, $\overline{\mu}_{ii}$ AND $\tilde{\mu}_{ii}$ FOR VARIANT IIb

Method	Parameter	i			
		I	2	3	4
Initial value	λ_i	0.8	0.6	I.5	0.4
First method, first approximation	$\overline{\lambda}_i$	0.930	0.575	I.700	0.072
	$\overline{\mu}_{ii}$	0.29243	0.422I3	0.05732	0.060I0
	$\tilde{\mu}_{ii}$	0.30932	0.42278	0.09750	0.I6792
First method, Second approximation	$\overline{\lambda}_i$	0.885	0.434	I.788	0.I06
	$\overline{\mu}_{ii}$	02I736	0.I5448	0.08079	0.0404I
	$\tilde{\mu}_{ii}$	0.22460	0.I8209	0.I6345	0.I2697
Second method, first approximation	$\overline{\lambda}_i$	0.779	0.645	I.507	0.408
	$\overline{\mu}_{ii}$	0.02657	0.0I946	0.0535I	0.0337I
	$\tilde{\mu}_{ii}$	0.02702	0.02I5I	0.05356	0.03378
Second method, second approximation	$\overline{\lambda}_i$	0.776	0.63I	I.5I9	0.400
	$\overline{\mu}_{ii}$	0.0I940	0.0II58	0.03858	0.026II
	$\tilde{\mu}_{ii}$	0.0I995	0.0I256	0.03894	0.02620
Method of least squares	$\overline{\lambda}_i$	0.734	0.559	I.395	0.350
	$\overline{\mu}_{ii}$	0.0I8I0	0.00774	0.026I7	0.0I657
	$\tilde{\mu}_{ii}$	0.02243	0.00940	0.037I9	0.0I902
More accurate least squares method, Second approximation	$\overline{\lambda}_i$	0.928	0.550	I.587	0.345
	$\overline{\mu}_{ii}$	0.02703	0.03296	0.03984	0.05354
	$\tilde{\mu}_{ii}$	0.0434I	0.03544	0.04743	0.05655

comparable to the accuracy of the first approximation using the second method, which confirms the theoretical results. The second method becomes more efficient for smaller λ. However, when the second approximation is taken into account (this case was not studied theoretically), the applicability of the first method can be extended to smaller values of λ. Thus, for example, for $\lambda = 1,2$, in the first approximation, the variances for both

TABLE 5. VALUES OF $\bar{\lambda}_i$, $\bar{\mu}_{ii}$ AND $\tilde{\mu}_{ii}$ FOR VARIANT IIIb

Method	Parameter	i = I	2	3	4
Initial value	λ	2.400	I.800	4.500	I,200
First method, first approximation	$\bar{\lambda}_i$	2.488	I.939	4.630	I.445
	$\bar{\mu}_{ii}$	0.072I2	0.06927	0.0629I	0.06223
	$\tilde{\mu}_{ii}$	0.07994	0.08868	0.082I3	0.02208
First method, second approximation	$\bar{\lambda}_i$	2.362	I.780	4.507	I.246
	$\bar{\mu}_{ii}$	0.04347	0.0449I	0.05395	0.02I7I
	$\tilde{\mu}_{ii}$	0.04990	0.0453I	0.05400	0.02383
Second method, first approximation	$\bar{\lambda}_i$	2.389	I.768	4.53I	I.I76
	$\bar{\mu}_{ii}$	0.05535	0.05249	0.06648	0.0339I
	$\tilde{\mu}_{ii}$	0.05547	0.0535I	0.06742	0.03449
Second method, second approximation	$\bar{\lambda}_i$	2.375	I.769	4.530	I.I96
	$\bar{\mu}_{ii}$	0.04680	0.0446[illegible]	0.05573	0.02302
	$\tilde{\mu}_{ii}$	0.04744	0.04578	0.05664	0.02304
Method of least squares	$\bar{\lambda}_i$	2.I20	I.603	4.033	I.063
	$\bar{\mu}_{ii}$	0.03503	0.030I8	0.06844	0.0254I
	$\tilde{\mu}_{ii}$	0.07490	0.05866	0.306I2	0.03987
Most accurate least squares method, second approximation	$\bar{\lambda}_i$	2.342	I.735	4.5I8	I.I27
	$\tilde{\mu}_{ii}$	0.05I75	0.04988	0.05696	0.02I86
	$\bar{\mu}_{ii}$	0.055I4	0.05404	0.0572[illegible]	0.027I2

methods differ considerably, but are nearly equal in the second approximation.

For still smaller λ (Table 4), the second method becomes more efficient in every respect -- even in the second approximation.

4. The method of least squares gives a biased estimate. For small λ, the improved least squares method in the second approximation is much worse than the second method in the second

TABLE 6. VALUES OF $\bar{\lambda}_i$, $\bar{\mu}_{ii}$, $\tilde{\mu}_{ii}$ FOR VARIANTS Ia, Ib, Ic (SECOND METHOD, FIRST AND SECOND APPROXIMATION)

Variant	Method	Parameter	i; λ_i			
			1; 0.270	2; 0.200	3; 0.500	4; 0.130
I a)	First approximation	$\bar{\lambda}_i$	0,184	0.173	0.523	0.131
		$\bar{\mu}_{ii}$	0.01343	0.01440	0.02122	0.01244
		$\tilde{\mu}_{ii}$	0.02083	0.01511	0.02173	0,01244
	Second approximation	$\bar{\lambda}_i$	0.201	0.188	0.542	0.137
		$\bar{\mu}_{ii}$	0,01447	0.01614	0.01989	0.01362
		$\tilde{\mu}_{ii}$	0.01924	0.01628	0.02166	0.01367
I b)	First approximation	$\bar{\lambda}_i$	0.275	0.200	0.530	0.133
		$\bar{\mu}_{ii}$	0.02518	0.02148	0.02130	0.01991
		$\tilde{\mu}_{ii}$	0,02520	0.02151	0,02280	0.01992
	Second approximation	$\bar{\lambda}_i$	0.278	0.195	0.543	0.137
		$\bar{\mu}_{ii}$	0,02454	0.02230	0.0[illegible]06	0,02006
		$\tilde{\mu}_{ii}$	0,02460	0.02232	0.02195	0.02011
I c)	First approximation	$\bar{\lambda}_i$	0.269	0.190	0,503	0.132
		$\bar{\mu}_{ii}$	0.00413	0.00360	0.00437	0.00217
		$\tilde{\mu}_{ii}$	0.00413	0.00370	0,00438	0,00216
	Second approximation	$\bar{\lambda}_i$	0.270	0.191	0,504	0,133
		$\bar{\mu}_{ii}$	0,00400	0.00343	0,00422	0.00210
		$\tilde{\mu}_{ii}$	0.00400	0.00351	0,00423	0.00211

approximation. For larger λ, these methods are practically comparable.

5. Taking into consideration the entire discussion and also the simplicity of the program, the second method in the second approximation is recommended for practical use. /30

TABLE 7. VALUES OF $\tilde{\mu}_{ij}$ FOR VARIANTS Ia, IIb, IIIc (SECOND METHOD, SECOND APPROXIMATION)

Variant	i \ j	I	2	3	4
I a)	I	0 0I924	-0,28	-0.2I	-0.3I
	2	-0.00500	0.0I628	-0.30	+0.09
	3	-0,0043I	-0.00563	0.02I66	-0.33
	4	-0.00502	0.00I35	-0.00573	0.0I367
II b)	I	0,0I995	-0,24	-0.I6	-0,37
	2	-0,00375	0,0I256	-0,2I	+0.II
	3	-0.00438	-0,00468	0,03894	-0,23
	4	-0.00843	0,00204	-0,00736	0,02620
III c)	I	0,00969	-0.II	0.22	-0,05
	2	-0.00I01	0,00868	0.I4	0
	3	0.00238	0.00I49	0.0I206	-0.I5
	4	-0.00036	0,00005	-0.00II2	0.00447

Note: The corresponding sample correlation coefficients are given above the diagonal.

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